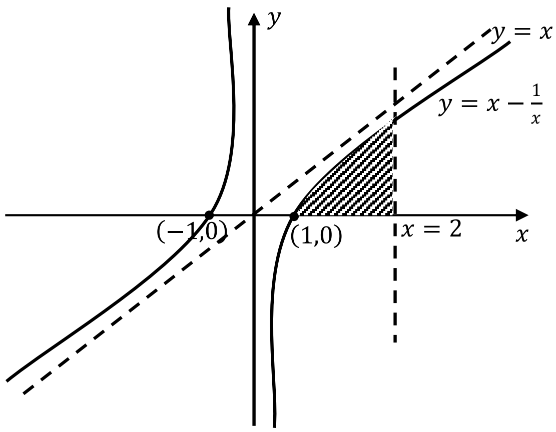
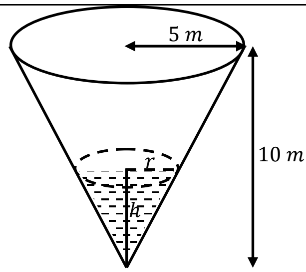
MARKING GUIDE
UMTA 425/1
2022

NO	SOLUTION	MKS	COMMENT
1	$y = \tan^{-1}\left(\frac{ax-b}{bx+a}\right)$ $\tan y = \frac{ax-b}{bx+a}$ $sec^2y\frac{dy}{dx} = \frac{(bx+a)\cdot a-(ax-b)\cdot b}{(bx+a)^2}$ $sec^2y\frac{dy}{dx} = \frac{a^2+b^2}{(bx+a)^2}$ But $sec^2y = 1 + tan^2y$ $= 1 + \left(\frac{ax-b}{bx+a}\right)^2$ $= \frac{(bx+a)^2+(ax-b)^2}{(bx+a)^2}$ $= \frac{b^2x^2+2abx+a^2+a^2x^2-2abx+b^2}{(bx+a)^2}$ $= \frac{b^2x^2+b^2+a^2+a^2x^2}{(bx+a)^2}$ $= \frac{b^2(1+x^2)+a^2(1+x^2)}{(bx+a)^2}$ $= \frac{(a^2+b^2)(1+x^2)}{(bx+a)^2}$ $\Rightarrow \frac{dy}{dx} = \frac{a^2+b^2}{(bx+a)^2}\cdot\frac{(bx+a)^2}{(a^2+b^2)(1+x^2)} = \frac{1}{1+x^2}$		
		05	
2	$\int_1^4 \frac{x^2+x}{\sqrt{2x+1}}dx$ Let $u = \sqrt{2x+1}$ $u^2 = 2x+1$ $2udu = 2dx$ $dx = udu$ $\Rightarrow x = \frac{u^2-1}{2}$ $x^2 = \frac{u^4-2u^2+1}{4}$		

	$\Rightarrow \int_{\sqrt{3}}^{3}\left(\frac{u^4-2u^2+1}{4}+\frac{u^2-1}{2}\right)du$ $=\frac{1}{4}\int_{\sqrt{3}}^{3}(u^4-1)\,du$ $=\frac{1}{4}\left[\frac{u^5}{5}-u\right]_{\sqrt{3}}^{3}$ $=\frac{1}{4}\left[\left(\frac{3^5}{5}-3\right)-\left(\frac{(\sqrt{3})^5}{5}-\sqrt{3}\right)\right]$ $=11.0535894$ $=11.0536$ (4 dps)		
		05	
3	L. H. S $=\frac{\sin x+\sin 3x+\sin 5x}{\cos x+\cos 3x+\cos 5x}$ $=\frac{\sin 5x+\sin x+\sin 3x}{\cos 5x+\cos x+\cos 3x}$ $=\frac{2\sin 3x\cos 2x+\sin 3x}{2\cos 3x\cos 2x+\cos 3x}$ $=\frac{\sin 3x(2\cos 2x+1)}{\cos 3x(2\cos 2x+1)}$ $=\tan 3x$ =R. H. S		
		05	
4	Let θ be the required angle For $x-3y+5=0$ $y=\frac{1}{3}x+\frac{5}{3}, m_1=\frac{1}{3}$ For $x+2y-1=0$ $y=-\frac{1}{2}x+\frac{1}{2}, m_2=-\frac{1}{2}$ From $\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$ $=\frac{\frac{1}{3}+\frac{1}{2}}{1+\frac{1}{3}\times-\frac{1}{2}}$ $=\frac{5/6}{5/6}=1$ $\therefore \theta=\tan^{-1}(1)=45^0$		
		05	

5	$y = x - \frac{1}{x}$ Vertical asymptotes , y −undefined $x = 0$ Slanting asymptote $y = x$ Intercepts $x;\ y = 0$ $0 = x^2 - 1$ $x = \pm 1;\ (-1,0)$ and $(1,0)$  A= $\int_1^2 \left(x - \frac{1}{x}\right) dx$ $A = \left[\frac{x^2}{2} - \ln x\right]_1^2$ $A = (2 - \ln 2) - \left(\frac{1}{2} - \ln 1\right)$ $A = 0.806852819$ $A = 0.8069$ sq. units		
		05	
6	$\sqrt{(3x - x)} - \sqrt{(7 + x)} = \sqrt{(16 + 2x)}$ $\sqrt{2x} - \sqrt{(7 + x)} = \sqrt{(16 + 2x)}$ Squaring both sides $2x - 2\sqrt{14x + 4x^2} + 7 + x = 16 + 2x$ $3x + 7 - 2\sqrt{14x + 4x^2} = 16 + 2x$		

	$-2\sqrt{14x+4x^2} = 9-x$ Squaring both sides again $4(14x+4x^2) = 81-18x+x^2$ $56x+16x^2 = 81-18x+x^2$ $15x^2+74x-81 = 0$ $(\quad)(\quad) = 0$		
		05	
7	No. of arrangement $= \frac{7!}{3!} = 840$ arrangements (MXMM)AIU No. of arrangement $= \frac{4!}{3!} \times 4! = 96$ arrangenents		
		05	
8	From distance $= \frac{\lvert ax_0+by_0+cz_0+d \rvert}{\sqrt{a^2+b^2+c^2}}$ $= \frac{\lvert 4(6)+3(-1)+5(2)-14 \rvert}{\sqrt{6^2+(-1)^2+2^2}}$ $= \frac{17}{\sqrt{41}}$ units		
		05	
9	(a) N (b) Let θ be the angle required $\boldsymbol{n} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \boldsymbol{d} = \begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$ $\boldsymbol{d} \cdot \boldsymbol{n} = \lvert\boldsymbol{d}\rvert\lvert\boldsymbol{n}\rvert \sin\theta$ $\begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \sqrt{9+16+144}\sqrt{1+4+1} \sin\theta$ $3-8+12 = \sqrt{169}\sqrt{6} \sin\theta$ $7 = 13\sqrt{6} \sin\theta$ $\theta = \sin^{-1}\left(\frac{7}{13\sqrt{6}}\right) = 12.7^0$		
		05	
10	(a)		



From similarities of figures

$\frac{H}{h} = \frac{R}{r}$

$\frac{10}{h} = \frac{5}{r}$

$r = \frac{h}{2}$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$

$\frac{dV}{dh} = \frac{\pi h^2}{4}$

$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$1 \cdot 5 = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{6}{\pi h^2}$

When $h = 4m$; $\frac{dh}{dt} = \frac{6}{\pi \times 16} = \frac{3}{8\pi} mmin^{-1}$

(b) Intercepts

$x; y = 0$

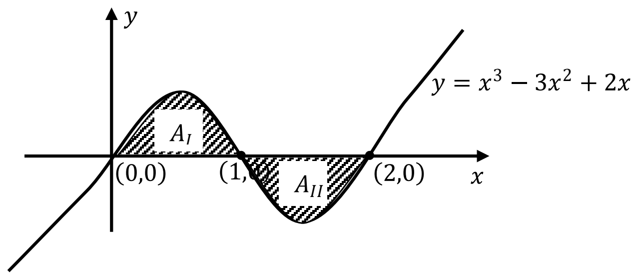
$0 = x(x-1)(x-2)$

$x = 0, x = 1, x = 2$

$\therefore$ (0,0), (1,0) and (2,0)

As $x \to +\infty, y \to +\infty$

As $x \to -\infty, y \to -\infty$

	 $A = A_I + A_{II}$ $A_1 = \int_0^1 (x^3 - 3x^2 + 2x)\, dx$ $A_I = \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1$ $A_I = \left(\frac{1}{4} - 1 + 1\right) - 0 = \frac{1}{4}$ sq. units $A_{II} = \int_0^1 (x^3 - 3x^2 + 2x)\, dx$ $A_{II} = \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2$ $A_{II} = (4 - 8 + 4) - \left(\frac{1}{4} - 1 + 1\right) = -\frac{1}{4}$ sq. units $\therefore A = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ sq. units		
		12	
11	$f(x) = \frac{x^4+x^3-6x^2-13x-6}{x^3-7x-6} = \frac{x^4+x^3-6x^2-13x-6}{(x+1)(x-3)(x+2)}$ Let $\frac{x^4+x^3-6x^2-13x-6}{(x+1)(x-3)(x+2)} \equiv Ax + B + \frac{C}{x+1} + \frac{D}{x-3} + \frac{E}{x+2}$ $x^4 + x^3 - 6x^2 - 13x - 6 \equiv (Ax + B)(x-3)(x+2)(x+1) + C(x-3)(x+2) +$ $D(x+1)(x+2) + E(x+1)(x-3)$ Put $x = 3; 81 + 27 - 54 - 39 - 6 = 20D$ $9 = 20D; \therefore D = \frac{9}{20}$ Put $x = -2; 16 - 8 - 24 + 26 - 6 = 5C$ $4 = 5E; \therefore E = \frac{4}{5}$ Put $x = -1; 1 - 1 - 6 + 13 - 6 = -4C$ $1 = -4C; \therefore C = -\frac{1}{4}$ Compare coefficients of; $x^4;\ 1 = A$		

	Put $x = 0$; $-6 = -6B - 6C + 2D - 3E$ $-6 = -6B - 6\left(-\frac{1}{4}\right) + 2\left(\frac{9}{20}\right) - 3\left(\frac{4}{5}\right)$ $-6 = -6B$; $\therefore B = 1$ $\therefore f(x) \equiv x + 1 - \frac{1}{4(x+1)} + \frac{9}{20(x-3)} + \frac{4}{5(x+2)}$ Hence $\int_4^5 f(x)\,dx = \int_4^5 (x+1)\,dx - \frac{1}{4}\int_4^5 \frac{1}{x+1}dx + \frac{9}{20}\int_4^5 \frac{1}{x-3}dx + \frac{4}{5}\int_4^5 \frac{1}{x+2}dx$ $= \left[\frac{x^2}{2} + x - \frac{1}{4}\ln(x+1) + \frac{9}{20}\ln(x-3) + \frac{4}{5}\ln(x+2)\right]_4^5 =$ $\left(\frac{5^2}{2} + 5 - \frac{1}{4}\ln(6) + \frac{9}{20}\ln(2) + \frac{4}{5}\ln(7)\right) - \left(\frac{4^2}{2} + 4 - \frac{1}{4}\ln(5) + \frac{9}{20}\ln(1) + \frac{4}{5}\ln(6)\right)$ $= 5.8896967$ $= 5.8897$ (4dps)		
		12	
12	(a) $\frac{dy}{dx} + \frac{2xy}{x^2+1} = x$ I.F $= e^{\int \frac{2x}{x^2+1}dx} = e^{\ln(x^2+1)} = x^2 + 1$ Multiplying through by $x^2 + 1$ gives $(x^2+1)\frac{dy}{dx} + 2xy = x^3 + x$ $\frac{d}{dx}\left((x^2+1)y\right) = x^3 + x$ $\int\left(\left((x^2+1)y\right)\right)dx = \int (x^3 + x)\,dx$ $\therefore y(x^2+1) = \frac{x^4}{4} + \frac{x^2}{2} + c$ (b) Let θ be the temperature of the liquid $\frac{d\theta}{dt} \propto (\theta - 22^0)$ $\frac{d\theta}{dt} = -k(\theta - 22^0)$ $\int \frac{d\theta}{\theta - 22^0} = -k\int dt$ $\ln(\theta - 22^0) = -kt + c$ $\theta - 22^0 = e^{-kt+c} = Ae^{-kt}$ $\theta - 22^0 = Ae^{-kt}$; $\theta = 22 + Ae^{-kt}$		

	At $t = 0, \theta = 100^0$C $100 = 22 + Ae^0; A = 78$ $\theta = 22 + 78e^{-kt}$ At $t = 1\ min, \theta = 92.2$ $92.2 = 22 + 78e^{-k}$ $70.2 = 78e^{-k}, k = \ln(^{10}/_9) = 0.10536$ $\theta = 22 + 78e^{-0.10536t}$ At $t = 5\ min, \theta = ?$ $\theta = 22 + 78e^{-0.10536\times5} = 63.45^0$C		
		12	
13	(a) $2^{2x+8} - 32(2^x) + 1$ $2^{2x} \cdot 2^8 - 32(2^x) + 1$ $(2^x)^2 \cdot 256 - 32(2^x) + 1 = 0$ Let $2^x = m$ $256m^2 - 32m + 1 = 0$ $(16m - 1)^2 = 0$ $m = \frac{1}{16} = 2^{-4}$ But $m = 2^x, 2^x = 2^{-4}$; $\therefore x = -4$ (b) $\log_a bc = x, \log_b ac = y, \log_c ab = z$ $bc = a^x$ ....(i) $ac = b^y$ ....(ii) $ab = c^z$ ....(iii) From eqn (i), $c = \frac{a^x}{b}$ From (ii); $a \cdot \frac{a^x}{b} = b^y$ $a^{1+x} = b^{1+y}; a = b^{\left(\frac{1+y}{1+x}\right)}$ From (iii), $b^{\left(\frac{1+y}{1+x}\right)} \cdot b = \left(\frac{a^x}{b}\right)^z$ $b^{\frac{1+y}{1+x}+1} = \frac{a^{xz}}{b^z}$		

	$b^{\left(\frac{1+y+1+x}{1+x}\right)+z} = \left(b^{\left(\frac{1+y}{1+x}\right)}\right)^{xz}$ $1 + y + 1 + x + z + xz = xz + xyz$ $2 + x + y + z = xyz$ $\therefore x + y + z = xyz = 2$		
		12	
14	(a) $(1-3x)^{1/3} = 1 + \frac{1}{3}(-3x) + \frac{\frac{1}{3}\times-\frac{2}{3}\times(-3x)^2}{2!} + \frac{\frac{1}{3}\times-\frac{2}{3}\times-\frac{5}{3}\times(-3x)^3}{3!} + \frac{\frac{1}{3}\times-\frac{2}{3}\times-\frac{5}{3}\times-\frac{8}{3}\times(-3x)^4}{4!} + \ldots$ $(1-3x)^{\frac{1}{3}} = 1 - x - x^2 - \frac{5}{3}x^3 - \frac{10}{3}x^4 + \ldots$ Putting $x = \frac{1}{8}$, $\left(1-\frac{3}{8}\right)^{1/3} \approx 1 - \frac{1}{8} - \left(\frac{1}{8}\right)^2 - \frac{5}{3}\left(\frac{1}{8}\right)^3 - \frac{10}{3}\left(\frac{1}{8}\right)^4$ $\left(\frac{5}{8}\right)^{1/3} \approx \frac{5255}{6144}$ $\sqrt[3]{5} \approx \frac{2\times5255}{6144} = 1.710611979 \approx 1.71$ (2dps) (b) $\frac{x-2}{x-1} \leq \frac{x+2}{x+1}$ $\frac{x-2}{x-1} - \frac{x+2}{x+1} \leq 0$ $\frac{(x-2)(x+1)-(x+2)(x-1)}{(x-1)(x+1)} \leq 0$ $\frac{-2x}{(x-1)(x+1)} \leq 0$ $\frac{x}{(x-1)(x+1)} \geq 0$ Critical values, $x = 0$ Undefined values, $x = 1, x = -1$		

x	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$x > 1$
x	$-$	$-$	$+$	$+$
$(x-1)(x+1)$	$+$	$-$	$-$	$+$
$\frac{x}{(x-1)(x+1)}$	$-$	$+$	$-$	$+$

$\therefore -1 \leq x \leq 0, x \geq 1$

12

15 (a) $10sin^2x + 10\sin x \sin x - cos^2x = 2$

$20sin^2x - cos^2x = 2$

$10(1 - cos^2x) - cos^2x = 2$

$10 - 11cos^2x = 2$

$cos^2x = \frac{8}{11}$

$\cos x = \pm\frac{2\sqrt{2}}{\sqrt{11}}$

For $\cos x = \frac{2\sqrt{2}}{\sqrt{11}}$

$x = \cos^{-1}\left(\frac{2\sqrt{2}}{\sqrt{11}}\right)$

$= 31.5^0, 328.5^0$

For $\cos x = -\frac{2\sqrt{2}}{\sqrt{11}}$

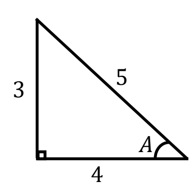
$x = \cos^{-1}\left(\frac{2\sqrt{2}}{\sqrt{11}}\right)$

$x = 148.5^0, 211.5^0$

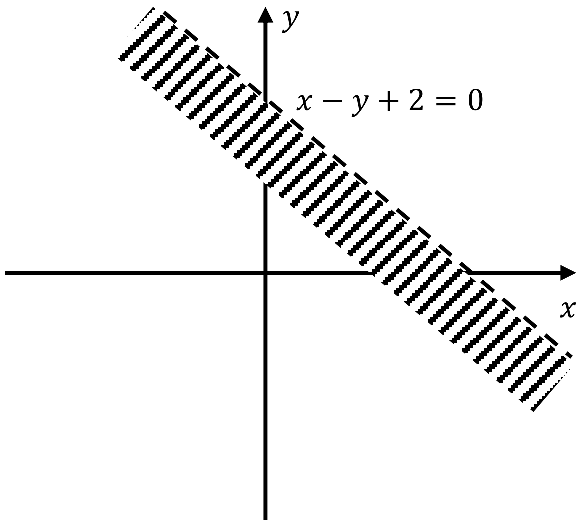
$\therefore x = \{31.5^0, 148.5^0, 211.5^0, 328.5^0\}$

(b) Let $\cos^{-1}\left(\frac{4}{5}\right) = A, \tan^{-1}\left(\frac{3}{5}\right) = B$

$\cos A = \frac{4}{5}, \tan B = \frac{3}{5}$



	$\tan A = \frac{3}{4}$ $\tan\left(\tan^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{3}{5}\right)\right) = \tan(A+B)$ $= \frac{\tan A + \tan B}{1 - \tan A \tan B}$ $= \frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}$ $= \frac{27}{11}$ $\therefore \cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{27}{11}\right)$		
		12	
16	(a) If $1+i$ is a root, the $1-i$ its conjugate is also a root. Sum of roots $= 1+i+1-i = 2$ Product of roots $= (1+i)(1-i) = 1^2+1^2 = 2$ ⇨ $z^2 - 2z + 2 = 0$ Long division of $z^4 - 6z^3 + 23z^2 - 34z + 26$ by $z^2 - 2z + 2$, quotient $z^2 - 4z + 13$: $z^4 - 6z^3 + 23z^2 - 34z + 26$ $-\ (z^4 - 2z^3 + 2z^2)$ $-4z^3 + 21z^2 + 2z^2 - 43z + 26$ $-\ (-4z^3 + 8z^2 - 8z)$ $13z^2 - 26z + 26$ $-\ (13z^2 - 26z + 26)$ $-\quad -\quad -$ ⇨ $z^2 - 4z + 13 = 0$ $z = \frac{4\pm\sqrt{(-4)^2-4\times1\times13}}{2\times1}$ $z = \frac{4\pm\sqrt{-36}}{2} = \frac{4\pm6i}{2} = 2\pm3i$ $\therefore$ The other roots are $1-i, 2+3i$ and $2-3i$ (b) Let $z = x+iy$ $\lvert x+iy+1-4i\rvert > \lvert x+iy-2-i\rvert$ $\lvert(x+1)+i(y-4)\rvert > \lvert(x-2)+i(y-1)\rvert$		

	$\sqrt{(x+1)^2+(y-4)^2} > \sqrt{(x-2)^2+(y-1)^2}$ Squaring both sides $(x+1)^2+(y-4)^2 > (x-2)^2+(y-1)^2$ $x^2+2x+1+y^2-8y+16 > x^2-4x+4+y^2-2y+1$ $x-y+2>0$  		
		12	